

2-LOCAL DERIVATIONS ON MATRIX RINGS OVER ASSOCIATIVE RINGS

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ABSTRACT. Let $M_n(\mathfrak{R})$ be the matrix ring over an associative ring \mathfrak{R} . In the present paper we prove that every inner 2-local derivation on the whole $M_n(\mathfrak{R})$ is an inner derivation if and only if every inner 2-local derivation on a certain subring of $M_n(\mathfrak{R})$, isomorphic to $M_2(\mathfrak{R})$, is an inner derivation. In particular, we prove that every inner 2-local derivation on the matrix ring $M_n(\mathfrak{R})$ over a commutative associative ring \mathfrak{R} is an inner derivation.

2000 Mathematics Subject Classification: Primary 16W25, 46L57; Secondary 47B47

INTRODUCTION

The present paper is devoted to 2-local derivations on associative rings. Recall that a 2-local derivation is defined as follows: given a ring \mathfrak{R} , a map $\Delta : \mathfrak{R} \rightarrow \mathfrak{R}$ (not additive in general) is called a 2-local derivation if for every $x, y \in \mathfrak{R}$, there exists a derivation $D_{x,y} : \mathfrak{R} \rightarrow \mathfrak{R}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

In 1997, P. Šemrl [5] introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H . A similar description for the finite-dimensional case appeared later in [3]. In the paper [4] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. In [2] the authors suggested a new technique and have generalized the above mentioned results of [5] and [3] for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation. In [1] we extended the above results and give a short proof of the theorem for arbitrary semi-finite von Neumann algebras.

In this article we develop an algebraic approach to the investigation of derivations and 2-local derivations on associative rings. Since we consider a sufficiently general case of associative rings we restrict our attention only on inner derivations and inner 2-local derivations. Namely, we consider the following problem: if an inner 2-local derivation on an associative ring is a derivation then is the latter derivation inner? The answer to this question is affirmative if the ring is generated by two elements (Proposition 10).

In this article we consider 2-local derivations on the matrix ring $M_n(\mathfrak{R})$ over an associative ring \mathfrak{R} . The first step of the investigation consists of proving that, if every inner 2-local derivation on a certain subring of the ring $M_n(\mathfrak{R})$, isomorphic to $M_2(\mathfrak{R})$, is an inner derivation then every inner 2-local derivation on the whole ring

Date: February 18, 2013.

Key words and phrases. derivation, inner derivation, 2-local derivation, matrix ring over an associative ring.

$M_n(\mathfrak{R})$ is also an inner derivation. In the case of a commutative associative ring \mathfrak{R} we prove that arbitrary inner 2-local derivation on $M_n(\mathfrak{R})$ is an inner derivation. The latter result extends the result of [4] to the infinite dimensional but commutative ring \mathfrak{R} .

The second step consists of proving that if every inner 2-local derivation on $M_n(\mathfrak{R})$ is an inner derivation then each inner 2-local derivation on a certain subring of the matrix ring $M_n(\mathfrak{R})$, isomorphic to $M_2(\mathfrak{R})$, is also an inner derivation.

1. 2-LOCAL DERIVATIONS ON MATRIX RINGS

Let \mathfrak{R} be a ring. Recall that a map $D : \mathfrak{R} \rightarrow \mathfrak{R}$ is called a derivation, if $D(x + y) = D(x) + D(y)$ and $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in \mathfrak{R}$. A derivation D on a ring \mathfrak{R} is called an inner derivation, if there exists an element $a \in \mathfrak{R}$ such that

$$D(x) = ax - xa, x \in A.$$

A map $\Delta : \mathfrak{R} \rightarrow \mathfrak{R}$ is called a 2-local derivation, if for any two elements $x, y \in \mathfrak{R}$ there exists a derivation $D_{x,y} : \mathfrak{R} \rightarrow \mathfrak{R}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

A map $\Delta : \mathfrak{R} \rightarrow \mathfrak{R}$ is called an inner 2-local derivation, if for any two elements $x, y \in \mathfrak{R}$ there exists an element $a \in \mathfrak{R}$ such that $\Delta(x) = ax - xa$, $\Delta(y) = ay - ya$.

Let \mathfrak{R} be an associative unital ring, $M_n(\mathfrak{R})$, $n > 1$, be the matrix ring over the associative ring \mathfrak{R} . Let $\bar{M}_2(\mathfrak{R})$ be a subring of $M_n(\mathfrak{R})$, generated by the subsets $\{e_{ii}M_{n+1}(\mathfrak{R})e_{jj}\}_{i,j=1}^2$ in $M_n(\mathfrak{R})$. It is clear that

$$\bar{M}_2(\mathfrak{R}) \cong M_2(\mathfrak{R}).$$

The following theorem is the main result of the paper.

Theorem 1. *Let \mathfrak{R} be an associative unital ring, and let $M_n(\mathfrak{R})$ be the matrix ring over \mathfrak{R} , $n > 1$. Then*

- 1) *every inner 2-local derivation on the matrix ring $M_n(\mathfrak{R})$ is an inner derivation if and only if every inner 2-local derivation on its subring $\bar{M}_2(\mathfrak{R})$ is an inner derivation,*
- 2) *if the ring \mathfrak{R} is commutative then every inner 2-local derivation on the matrix ring $M_n(\mathfrak{R})$ is an inner derivation*

First let us prove lemmata and propositions which are necessary for the proof of theorem 1.

Let \mathfrak{R} be an associative unital ring, and let $\{e_{ij}\}_{i,j=1}^n$ be the set of matrix units in $M_n(\mathfrak{R})$ such that e_{ij} is a $n \times n$ -dimensional matrix in $M_n(\mathfrak{R})$, i.e. $e_{ij} = (a_{kl})_{k,l=1}^n$, the (i, j) -th component of which is 1 (the unit of \mathfrak{R}), i.e. $a_{ij} = 1$, and the rest components are zeros.

Let $\Delta : M_n(\mathfrak{R}) \rightarrow M_n(\mathfrak{R})$ be an inner 2-local derivation. Consider the subset $\{a(ij)\}_{i,j=1}^n \subset M_n(\mathfrak{R})$ such that

$$\Delta(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij).$$

Put $a_{ij} = e_{ii}a(ji)e_{jj}$, for all pairs of different indices i, j and let $\{a_{kl}\}_{k \neq l}$ be the set of all such elements.

Lemma 2. *Let $\Delta : M_n(\mathfrak{R}) \rightarrow M_n(\mathfrak{R})$ be an inner 2-local derivation. Then for any pair i, j of different indices the following equality holds*

$$\Delta(e_{ij}) = \{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l} + a(ij)e_{ii}e_{ij} - e_{ij}a(ij)e_{jj}, \quad (1)$$

where $a(ij)_{ii}$, $a(ij)_{jj}$ are components of the matrices $e_{ii}a(ij)e_{ii}$, $e_{jj}a(ij)e_{jj}$.

Proof. Let m be an arbitrary index different from i, j and let $a(ij, ik) \in M_n(\mathfrak{R})$ be an element such that

$$\Delta(e_{im}) = a(ij, im)e_{im} - e_{im}a(ij, im) \text{ and } \Delta(e_{ij}) = a(ij, im)e_{ij} - e_{ij}a(ij, im).$$

We have

$$\Delta(e_{im}) = a(ij, im)e_{im} - e_{im}a(ij, im) = a(im)e_{im} - e_{im}a(im)$$

and

$$e_{mm}a(ij, im)e_{ij} = e_{mm}a(im)e_{ij}.$$

Then

$$\begin{aligned} e_{mm}\Delta(e_{ij})e_{jj} &= e_{mm}(a(ij, im)e_{ij} - e_{ij}a(ij, im))e_{jj} = \\ e_{mm}a(ij, im)e_{ij} - 0 &= e_{mm}a(im)e_{ij} - e_{mm}e_{ij}\{a_{kl}\}_{k \neq l}e_{jj} = \\ e_{mm}a_{mi}e_{ij} - e_{mm}e_{ij}\{a_{kl}\}_{k \neq l}e_{jj} &= e_{mm}\{a_{kl}\}_{k \neq l}e_{ij} - e_{mm}e_{ij}\{a_{kl}\}_{k \neq l}e_{jj} = \\ e_{mm}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{jj}. \end{aligned}$$

Similarly,

$$\begin{aligned} e_{mm}\Delta(e_{ij})e_{ii} &= e_{mm}(a(ij, im)e_{ij} - e_{ij}a(ij, im))e_{ii} = \\ e_{mm}a(ij, im)e_{ij}e_{ii} - 0 &= 0 - 0 = e_{mm}\{a_{kl}\}_{k \neq l}e_{ij}e_{ii} - e_{mm}e_{ij}\{a_{kl}\}_{k \neq l}e_{ii} = \\ e_{mm}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{ii}. \end{aligned}$$

Let $a(ij, mj) \in M_n(\mathfrak{R})$ be an element such that

$$\Delta(e_{mj}) = a(ij, mj)e_{mj} - e_{mj}a(ij, mj) \text{ and } \Delta(e_{ij}) = a(ij, mj)e_{ij} - e_{ij}a(ij, mj).$$

We have

$$\Delta(e_{mj}) = a(ij, mj)e_{mj} - e_{mj}a(ij, mj) = a(mj)e_{mj} - e_{mj}a(mj).$$

and

$$e_{ij}a(ij, mj)e_{mm} = e_{ij}a(mj)e_{mm}.$$

Then

$$\begin{aligned} e_{ii}\Delta(e_{ij})e_{mm} &= e_{ii}(a(ij, mj)e_{ij} - e_{ij}a(ij, mj))e_{mm} = \\ 0 - e_{ij}a(ij, mj)e_{mm} &= 0 - e_{ij}a(mj)e_{mm} = 0 - e_{ij}a_{jm}e_{mm} = \\ e_{ii}\{a_{kl}\}_{k \neq l}e_{ij}e_{mm} - e_{ij}\{a_{kl}\}_{k \neq l}e_{mm} &= \\ e_{ii}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{mm}. \end{aligned}$$

Also we have

$$\begin{aligned} e_{jj}\Delta(e_{ij})e_{mm} &= e_{jj}(a(ij, mj)e_{ij} - e_{ij}a(ij, mj))e_{mm} = \\ 0 - 0 &= e_{jj}\{a(ij)\}_{i \neq j}e_{ij}e_{mm} - e_{jj}e_{ij}\{a(ij)\}_{i \neq j}e_{mm} = \\ e_{jj}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{mm}, \end{aligned}$$

$$\begin{aligned} e_{ii}\Delta(e_{ij})e_{ii} &= e_{ii}(a(ij)e_{ij} - e_{ij}a(ij))e_{ii} = \\ 0 - e_{ij}a(ij)e_{ii} &= 0 - e_{ij}a(ij)e_{ii} = 0 - e_{ij}a_{ji}e_{ii} = \\ e_{ii}\{a_{kl}\}_{k \neq l}e_{ij}e_{ii} - e_{ij}\{a_{kl}\}_{k \neq l}e_{ii} &= \\ e_{ii}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{ii}. \end{aligned}$$

$$\begin{aligned} e_{jj}\Delta(e_{ij})e_{jj} &= e_{jj}(a(ij)e_{ij} - e_{ij}a(ij))e_{jj} = \\ e_{jj}a(ij)e_{ij} - 0 &= e_{jj}a_{ji}e_{ij} - 0 = \\ e_{jj}\{a_{kl}\}_{k \neq l}e_{ij} - e_{jj}e_{ij}\{a_{kl}\}_{k \neq l}e_{jj} &= \end{aligned}$$

$$e_{jj}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{jj}.$$

Thus

$$\begin{aligned} \Delta(e_{ij}) &= \sum_{k,l=1}^n e_{kk}\Delta(e_{ij})e_{ll} = \\ &= \sum_{k,l=1}^n e_{kk}(\{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l})e_{ll} + e_{ii}\Delta(e_{ij})e_{jj} = \\ &= \{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l} + a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}. \end{aligned}$$

The proof is complete. \triangleright

Consider the element $x = \sum_{i=1}^{n-1} e_{i,i+1}$. For arbitrary different indices i and j there exists an element $c \in M_n(A)$ such that

$$\Delta(e_{ij}) = ce_{ij} - e_{ij}c \text{ and } \Delta(x) = cx - xc.$$

Let $c = \sum_{i,j=1}^n c_{ij}$ be the Pierce decomposition of c , $a_{ii} = c_{ii}$ for any i and $\bar{a} = \sum_{i,j=1}^n a_{ij}$.

Lemma 3. *Let $\Delta : M_n(\mathfrak{R}) \rightarrow M_n(\mathfrak{R})$ be an inner 2-local derivation. Suppose that the ring \mathfrak{R} is commutative and k, l are arbitrary different indices. If $b \in M_n(\mathfrak{R})$ is an element such that*

$$\Delta(e_{kl}) = be_{kl} - e_{kl}b \text{ and } \Delta(x) = bx - xb$$

then $c_{kk} - c_{ll} = b_{kk} - b_{ll}$.

Proof. We can suppose that $k < l$. We have

$$\Delta(x) = cx - xc = bx - xb.$$

Hence

$$e_{kk}(cx - xc)e_{k+1,k+1} = e_{kk}(bx - xb)e_{k+1,k+1}$$

and

$$c_{kk} - c_{k+1,k+1} = b_{kk} - b_{k+1,k+1}.$$

Then for the sequence

$$(k, k+1), (k+1, k+2), \dots, (l-1, l)$$

we have

$$c_{kk} - c_{k+1,k+1} = b_{kk} - b_{k+1,k+1}, c_{k+1,k+1} - c_{k+2,k+2} = b_{k+1,k+1} - b_{k+2,k+2}, \dots$$

$$c_{l-1,l-1} - c_{ll} = b_{l-1,l-1} - b_{ll}.$$

Hence

$$c_{kk} - b_{kk} = c_{k+1,k+1} - b_{k+1,k+1}, c_{k+1,k+1} - b_{k+1,k+1} = c_{k+2,k+2} - b_{k+2,k+2}, \dots$$

$$c_{l-1,l-1} - b_{l-1,l-1} = c_{ll} - b_{ll}.$$

Therefore $c_{kk} - b_{kk} = c_{ll} - b_{ll}$, i.e. $c_{kk} - c_{ll} = b_{kk} - b_{ll}$. The proof is complete. \triangleright

Proposition 4. *Let $\Delta : M_n(\mathfrak{R}) \rightarrow M_n(\mathfrak{R})$ be an inner 2-local derivation, $n > 1$. Then*

1) *if every inner 2-local derivation on the ring $\bar{M}_2(\mathfrak{R})$ is an inner derivation then every inner 2-local derivation on $M_n(\mathfrak{R})$ is an inner derivation.*

2) *if the ring \mathfrak{R} is commutative then every inner 2-local derivation on $M_n(\mathfrak{R})$ is an inner derivation.*

Proof. Let x be a symmetric matrix in $M_n(\mathfrak{R})$ and let $d(ij) \in M_n(\mathfrak{R})$ be an element such that

$$\Delta(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) \text{ and } \Delta(x) = d(ij)x - xd(ij)$$

and $i \neq j$. Then by Lemma 2

$$\begin{aligned} \Delta(e_{ij}) &= d(ij)e_{ij} - e_{ij}d(ij) = \\ e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} + (1 - e_{ii})d(ij)e_{ij} - e_{ij}d(ij)(1 - e_{jj}) &= \\ a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} + \{a_{kl}\}_{k \neq l}e_{ij} - e_{ij}\{a_{kl}\}_{k \neq l} \end{aligned}$$

for all i, j .

Since $e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} = a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}$ we have

$$(1 - e_{ii})d(ij)e_{ii} = \{a_{kl}\}_{k \neq l}e_{ii}, e_{jj}d(ij)(1 - e_{jj}) = e_{jj}\{a_{kl}\}_{k \neq l}$$

for all different i and j .

Hence by lemma 3 we have

$$\begin{aligned} e_{jj}\Delta(x)e_{ii} &= e_{jj}(d(ij)x - xd(ij))e_{ii} = \\ e_{jj}d(ij)(1 - e_{jj})xe_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}x(1 - e_{ii})d(ij)e_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} &= \\ e_{jj}\{a_{kl}\}_{k \neq l}xe_{ii} - e_{jj}x\{a_{kl}\}_{k \neq l}e_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii}. \end{aligned}$$

Similarly

$$\begin{aligned} e_{ii}\Delta(x)e_{jj} &= \\ e_{ii}\{a_{kl}\}_{k \neq l}xe_{jj} - e_{ii}x\{a_{kl}\}_{k \neq l}e_{jj} + e_{ii}d(ij)e_{ii}xe_{jj} - e_{ii}xe_{jj}d(ij)e_{jj}. \end{aligned}$$

Also, we have

$$\begin{aligned} e_{ii}\Delta(x)e_{ii} &= e_{ii}(d(ij)x - xd(ij))e_{ii} = \\ e_{ii}d(ij)(1 - e_{ii})xe_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}x(1 - e_{ii})d(ij)e_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} &= \\ e_{ii}\{a_{kl}\}_{k \neq l}xe_{ii} - e_{ii}x\{a_{kl}\}_{k \neq l}e_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} \end{aligned}$$

and

$$\begin{aligned} e_{jj}\Delta(x)e_{jj} &= e_{jj}(d(ij)x - xd(ij))e_{jj} = \\ e_{jj}\{a_{kl}\}_{k \neq l}xe_{jj} - e_{jj}x\{a_{kl}\}_{k \neq l}e_{jj} + e_{jj}d(ij)e_{jj}xe_{jj} - e_{jj}xe_{jj}d(ij)e_{jj}. \end{aligned}$$

Now in both cases 1) and 2) we have

$$\begin{aligned} e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} &= c_{ii}e_{ii}xe_{ii} - e_{ii}xe_{ii}c_{ii}, \\ e_{jj}d(ij)e_{jj}xe_{jj} - e_{jj}xe_{jj}d(ij)e_{jj} &= c_{jj}e_{jj}xe_{jj} - e_{jj}xe_{jj}c_{jj}, \\ e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} &= c_{jj}e_{jj}xe_{ii} - e_{jj}xe_{ii}c_{ii}, \\ e_{ii}d(ij)e_{ii}xe_{jj} - e_{ii}xe_{jj}d(ij)e_{jj} &= c_{ii}e_{ii}xe_{jj} - e_{ii}xe_{jj}c_{jj}. \end{aligned}$$

Hence

$$\begin{aligned} \Delta(x) &= \left(\sum_i c_{ii}\right)x - x\left(\sum_i c_{ii}\right) + \{a_{kl}\}_{k \neq l}x - x\{a_{kl}\}_{k \neq l} = \\ &\quad \bar{a}x - x\bar{a} \end{aligned}$$

for all $x \in M_n(\mathfrak{R})$. The proof is complete. \triangleright

2. EXTENSION OF DERIVATIONS AND 2-LOCAL DERIVATIONS

Proposition 5. *Let $M_2(\mathfrak{R})$ be the matrix ring over a unital associative ring \mathfrak{R} and let $D : e_{11}M_2(\mathfrak{R})e_{11} \rightarrow e_{11}M_2(\mathfrak{R})e_{11}$ be a derivation on the subring $e_{11}M_2(\mathfrak{R})e_{11}$. Then, if $\phi : e_{11}M_2(\mathfrak{R})e_{11} \rightarrow e_{22}M_2(\mathfrak{R})e_{22}$ is an isomorphism defined as $\phi(a) = e_{21}ae_{12}$, $a \in e_{11}M_2(\mathfrak{R})e_{11}$ then the map defined by the following conditions*

- 1) $\bar{D}(a) = D(a)$, $a \in e_{11}M_2(\mathfrak{R})e_{11}$,
 - 2) $\bar{D}(a) = \phi \circ D \circ \phi^{-1}(a)$, $a \in e_{22}M_2(\mathfrak{R})e_{22}$,
 - 3) $\bar{D}(e_{12}) = e_{12}$, $\bar{D}(e_{21}) = -e_{21}$,
 - 4) $\bar{D}(a) = D(ae_{21})e_{12} + ae_{21}D(e_{12})$, $a \in e_{11}M_2(\mathfrak{R})e_{22}$,
 - 5) $\bar{D}(a) = D(e_{21})e_{12}a + e_{21}D(e_{12}a)$, $a \in e_{22}M_2(\mathfrak{R})e_{11}$,
 - 6) $\bar{D}(a) = \bar{D}(e_{11}ae_{11}) + \bar{D}(e_{11}ae_{22}) + \bar{D}(e_{22}ae_{11}) + \bar{D}(e_{22}ae_{22})$, $a \in M_2(\mathfrak{R})$,
- is a derivation.*

Proof. For every $a \in M_2(\mathfrak{R})$ the value $\bar{D}(a)$ is uniquely defined. Therefore \bar{D} is a map.

It is clear that \bar{D} is additive. Now we will prove that $\bar{D}(ab) = \bar{D}(a)b + a\bar{D}(b)$ for arbitrary elements $a, b \in M_2(\mathfrak{R})$.

Let $a_1 = e_{11}ae_{11}$, $a_{12} = e_{11}ae_{22}$, $a_{21} = e_{22}ae_{11}$, $a_2 = e_{22}ae_{22}$, $b_1 = e_{11}be_{11}$, $b_{12} = e_{11}be_{22}$, $b_{21} = e_{22}be_{11}$, $b_2 = e_{22}be_{22}$, $\bar{D}^\perp = \phi \circ D \circ \phi^{-1}$ for arbitrary elements $a, b \in M_2(\mathfrak{R})$. Then we have the following Pierce decompositions of the elements a and b

$$a = a_1 + a_{12} + a_{21} + a_2, b = b_1 + b_{12} + b_{21} + b_2.$$

The following equalities hold

$$\begin{aligned} \bar{D}(a_1b_1) &= D(a_1b_1), \\ \bar{D}(a_1b_2) &= \bar{D}(0) = 0 = \bar{D}(a_1)b_2 + a_1\bar{D}(b_2), \\ \bar{D}(a_1b_{12}) &= D(a_1b_{12}e_{21})e_{12} + a_1b_{12}e_{21}\bar{D}(e_{12}) = \\ &= D(a_1)b_{12} + a_1D(b_{12}e_{21})e_{12} + a_1b_{12}e_{21}\bar{D}(e_{12}) = \\ &= D(a_1)b_{12} + a_1(D(b_{12}e_{21})e_{12} + b_{12}e_{21}\bar{D}(e_{12})) = D(a_1)b_{12} + a_1\bar{D}(b_{12}) = \\ &= \bar{D}(a_1)b_{12} + a_1\bar{D}(b_{12}), \\ \bar{D}(a_1b_{21}) &= \bar{D}(0) = 0 = a_1(\bar{D}(e_{21})e_{12}b_{21} + e_{21}D(e_{12}b_{21})) = \\ &= a_1\bar{D}(b_{21}) = D(a_1)b_{21} + a_1\bar{D}(b_{21}) = \bar{D}(a_1)b_{21} + a_1\bar{D}(b_{21}), \\ \bar{D}(a_2b_1) &= \bar{D}(a_2)b_1 + a_2\bar{D}(b_1) = \bar{D}(a_2)b_1 + a_2D(b_1) = 0, \\ \bar{D}(a_{12}b_1) &= \bar{D}(0) = 0 = (D(a_{12}e_{21})e_{12} + a_{12}e_{21}\bar{D}(e_{12}))b_1 = \\ &= \bar{D}(a_{12})b_1 = \bar{D}(a_{12})b_1 + a_{12}D(b_1) = \bar{D}(a_{12})b_1 + a_{12}\bar{D}(b_1). \end{aligned}$$

Also, since

$$\begin{aligned} \bar{D}(e_{12})a_{12} + e_{12}\bar{D}(a_{21}) &= e_{12}a_{21} + e_{12}(\bar{D}(e_{21})e_{21}a_{21} + e_{21}D(e_{12}a_{21})) = \\ &= e_{12}a_{21} - e_{12}e_{21}e_{12}a_{21} + e_{12}e_{21}D(e_{12}a_{21}) = D(e_{12}a_{21}) \end{aligned}$$

we have

$$\begin{aligned} \bar{D}(a_{21}b_1) &= \bar{D}(e_{21})e_{12}a_{21}b_1 + e_{21}D(e_{12}a_{21}b_1) = \\ &= -a_{21}b_1 + e_{21}(D(e_{12}a_{21})b_1 + e_{12}a_{21}D(b_1)) = \\ &= -a_{21}b_1 + e_{21}D(e_{12}a_{21})b_1 + a_{21}D(b_1) = \\ &= -a_{21}b_1 + e_{21}(\bar{D}(e_{12})a_{21} + e_{12}\bar{D}(a_{21}))b_1 + a_{21}D(b_1) = \\ &= -a_{21}b_1 + a_{21}b_1 + e_{22}\bar{D}(a_{21})b_1 + a_{21}D(b_1) = \\ &= e_{22}\bar{D}(a_{21})b_1 + a_{21}D(b_1) = \bar{D}(a_{21})b_1 + a_{21}\bar{D}(b_1) \end{aligned}$$

by condition 5). Similarly we have

$$\begin{aligned}
\bar{D}(a_{21}b_{12}) &= \bar{D}^\perp(a_{21}e_{12}e_{21}b_{12}) = \\
&= \bar{D}^\perp(a_{21}e_{12})e_{21}b_{12} + a_{21}e_{12}\bar{D}^\perp(e_{21}b_{12}) = \\
&= \bar{D}^\perp(e_{21}e_{12}a_{21}e_{12})e_{21}b_{12} + a_{21}e_{12}\bar{D}^\perp(e_{21}b_{12}e_{21}e_{12}) = \\
&= \phi \circ D(e_{12}a_{21})e_{21}b_{12} + a_{21}e_{12}\phi \circ D(b_{12}e_{21}) = \\
&= e_{21}D(e_{12}a_{21})e_{12}e_{21}b_{12} + a_{21}e_{12}e_{21}D(b_{12}e_{21})e_{12} = \\
&= e_{21}D(e_{12}a_{21})b_{12} - a_{21}b_{12} + a_{21}b_{12} + a_{21}D(b_{12}e_{21})e_{12} = \\
&= e_{21}D(e_{12}a_{21})b_{12} + \bar{D}(e_{21})e_{12}a_{21}b_{12} + a_{21}b_{12}e_{21}\bar{D}(e_{12}) + a_{21}D(b_{12}e_{21})e_{12} = \\
&= \bar{D}(e_{21}D(e_{12}a_{21}) + \bar{D}(e_{21})e_{12}a_{21})b_{12} + a_{21}(b_{12}e_{21}\bar{D}(e_{12}) + D(b_{12}e_{21})e_{12}) = \\
&= \bar{D}(a_{21})b_{12} + a_{21}\bar{D}(b_{12})
\end{aligned}$$

and

$$\bar{D}(a_{12}b_{21}) = \bar{D}(a_{12})b_{21} + a_{12}\bar{D}(b_{21}).$$

By conditions 4) and 5) above the following equalities hold

$$\bar{D}(a_{12}b_{12}) = \bar{D}(a_{12})b_{12} + a_{12}\bar{D}(b_{12}) = 0,$$

$$\bar{D}(a_{21}b_{21}) = \bar{D}(a_{21})b_{21} + a_{21}\bar{D}(b_{21}) = 0.$$

By these equalities we have

$$\begin{aligned}
\bar{D}(ab) &= \bar{D}((a_1 + a_{12} + a_{21} + a_2)(b_1 + b_{12} + b_{21} + b_2)) = \\
&= \bar{D}(a_1b_1) + \bar{D}(a_1b_{12}) + \bar{D}(a_1b_{21}) + \bar{D}(a_1b_2) + \bar{D}(a_{12}b_1) + \dots \\
&\quad + \bar{D}(a_2b_{21}) + \bar{D}(a_2b_2) = \bar{D}(a)b + a\bar{D}(b).
\end{aligned}$$

Hence, the map \bar{D} is a derivation and it is an extension of the derivation D on the ring $M_2(\mathfrak{R})$. The proof is complete. \triangleright

Let $\bar{M}_m(\mathfrak{R})$ be a subring of $M_n(\mathfrak{R})$, $m < n$, generated by the subsets $\{e_{ii}M_n(\mathfrak{R})e_{jj}\}_{ij=1}^m$ in $M_n(\mathfrak{R})$. It is clear that

$$\bar{M}_m(\mathfrak{R}) \cong M_m(\mathfrak{R}).$$

Proposition 6. *Let \mathfrak{R} be an associative ring, and let $M_n(\mathfrak{R})$ be a matrix ring over \mathfrak{R} , $n > 2$. Then every derivation on $\bar{M}_2(\mathfrak{R})$ can be extended to a derivation on $M_n(\mathfrak{R})$.*

Proof. By proposition 5 every derivation on $\bar{M}_2(\mathfrak{R})$ can be extended to a derivation on $M_4(\mathfrak{R})$. In its turn, every derivation on $\bar{M}_4(\mathfrak{R})$ can be extended to a derivation on $M_8(\mathfrak{R})$ and so on. Thus every derivation ∂ on $\bar{M}_2(\mathfrak{R})$ can be extended to a derivation D on $M_{2^k}(\mathfrak{R})$. Suppose that $n \leq 2^k$. Let $e = \sum_{i=1}^n e_{ii}$ and

$$\bar{D}(a) = eD(a)e, a \in \bar{M}_n(\mathfrak{R}).$$

Then $\bar{D} : \bar{M}_n(\mathfrak{R}) \rightarrow \bar{M}_n(\mathfrak{R})$ and \bar{D} is a derivation on $\bar{M}_n(\mathfrak{R})$. Indeed, it is clear that \bar{D} is a linear map. At the same time, for all $a, b \in \bar{M}_n(\mathfrak{R})$ we have

$$\bar{D}(ab) = eD(ab)e = e(D(a)b + aD(b))e =$$

$$eD(a)be + eaD(b)e = eD(a)eb + aeD(b)e = \bar{D}(a)b + a\bar{D}(b).$$

Hence, $\bar{D} : \bar{M}_n(\mathfrak{R}) \rightarrow \bar{M}_n(\mathfrak{R})$ is a derivation. At the same time, on the subalgebra $\bar{M}_2(\mathfrak{R})$ the derivation \bar{D} coincides with the derivation ∂ . Therefore, \bar{D} is an extension of ∂ to $\bar{M}_n(\mathfrak{R})$. \triangleright

Thus, in the case of the ring $M_2(\mathfrak{R})$ for any derivation on the subring $e_{11}M_2(\mathfrak{R})e_{11}$ we can take its extension onto the whole $M_2(\mathfrak{R})$ defined as in proposition 5, which is also a derivation.

In proposition 7 we take the extensions of derivations defined as in proposition 5.

Proposition 7. *Let $M_2(\mathfrak{R})$ be the matrix ring over a unital associative ring \mathfrak{R} and let $\Delta : e_{11}M_2(\mathfrak{R})e_{11} \rightarrow e_{11}M_2(\mathfrak{R})e_{11}$ be a 2-local derivation on the subring $e_{11}M_2(\mathfrak{R})e_{11}$. Then, if $\phi : e_{11}M_2(\mathfrak{R})e_{11} \rightarrow e_{22}M_2(\mathfrak{R})e_{22}$ is an isomorphism defined as $\phi(a) = e_{21}ae_{12}$, $a \in e_{11}M_2(\mathfrak{R})e_{11}$ then the map ∇ defined by the following conditions is a 2-local derivation:*

- 1) $\nabla(a) = \Delta(a)$ if $a \in e_{11}M_2(\mathfrak{R})e_{11}$,
- 2) $\nabla(a) = \phi \circ \Delta \circ \phi^{-1}(a)$ if $a \in e_{22}M_2(\mathfrak{R})e_{22}$,
- 3) $\nabla(e_{12}) = e_{12}$, $\nabla(e_{21}) = -e_{21}$,
- 4) $\nabla(a) = \Delta(ae_{21})e_{12} + ae_{21}\nabla(e_{12})$ if $a \in e_{11}M_2(\mathfrak{R})e_{22}$,
- 5) $\nabla(a) = \nabla(e_{21})e_{12}a + e_{21}\Delta(e_{12}a)$ if $a \in e_{22}M_2(\mathfrak{R})e_{11}$,
- 6)

$$\nabla(a) = \bar{D}(e_{11}ae_{11}) + \bar{D}(e_{11}ae_{22}) + \bar{D}(e_{22}ae_{11}) + \bar{D}(e_{22}ae_{22}),$$

$a \in M_2(\mathfrak{R})$, where, if $e_{11}ae_{11} \neq 0$ then \bar{D} is the extension of the derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(e_{11}ae_{11}) = D(e_{11}ae_{11}),$$

if $e_{11}ae_{11} = 0$ and $e_{22}ae_{22} \neq 0$ then \bar{D} is the extension of the derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(e_{12}e_{22}ae_{22}e_{21}) = D(e_{12}e_{22}ae_{22}e_{21}),$$

if $e_{11}ae_{11} = e_{22}ae_{22} = 0$ and $e_{11}ae_{22} \neq 0$ then \bar{D} is the extension of the derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(e_{11}ae_{22}e_{21}) = D(e_{11}ae_{22}e_{21}),$$

if $e_{11}ae_{11} = e_{22}ae_{22} = e_{11}ae_{22} = 0$ and $e_{22}ae_{11} \neq 0$ then \bar{D} is the extension of the derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(e_{12}e_{22}ae_{11}) = D(e_{12}e_{22}ae_{11}).$$

Proof. It is clear that, if $a \in e_{11}M_2(\mathfrak{R})e_{11}$ then the value $\nabla(a)$ defined in the case 1) coincides with the value $\nabla(a)$ defined in the case 6). Similarly, if $a \in e_{22}M_2(\mathfrak{R})e_{22}$ then the value $\nabla(a)$ defined in the case 2) coincides with the value of $\nabla(a)$ defined in the case 6) and so on. Hence ∇ is a correctly defined map.

Now we should prove that ∇ is a 2-local derivation. Let a, b be arbitrary elements of the algebra $M_2(\mathfrak{R})$. Suppose that $e_{11}ae_{11} \neq 0$, $e_{11}be_{11} \neq 0$. Then by the definition there exists a derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(a) = D(e_{11}ae_{11}) \text{ and } \Delta(b) = D(e_{11}be_{11}).$$

Let \bar{D} be the extension of the derivation D satisfying the conditions of the proposition 5. Hence

$$\nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b)$$

by the definition of the map ∇ .

Now suppose that $e_{11}ae_{11} = 0$, $e_{22}ae_{22} \neq 0$ and $e_{11}be_{11} \neq 0$. Then by the definition there exists a derivation D on $e_{11}M_2(\mathfrak{R})e_{11}$ such that

$$\Delta(a) = D(e_{12}e_{22}ae_{22}e_{21}) \text{ and } \Delta(b) = D(e_{11}be_{11}).$$

Let \bar{D} be the extension of the derivation D satisfying the conditions of the proposition 5. Hence

$$\nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b)$$

and so on. In all cases there exists a derivation \bar{D} such that

$$\nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b)$$

Since a, b are arbitrary elements in $M_2(\mathfrak{R})$ we have ∇ is a 2-local derivation. \triangleright

Proposition 8. *Let \mathfrak{R} be an associative ring, and let $M_n(\mathfrak{R})$ be a matrix ring over \mathfrak{R} , $n > 2$. Then every 2-local derivation on $\bar{M}_2(\mathfrak{R})$ can be extended to a 2-local derivation on $M_n(\mathfrak{R})$.*

Proof. By proposition 7 every 2-local derivation on $\bar{M}_2(\mathfrak{R})$ can be extended to a 2-local derivation on $M_4(\mathfrak{R})$. In its turn, every 2-local derivation on $\bar{M}_4(\mathfrak{R})$ can be extended to a 2-local derivation on $M_8(\mathfrak{R})$ and so on. Thus every 2-local derivation Δ on $\bar{M}_2(\mathfrak{R})$ can be extended to a 2-local derivation $\bar{\Delta}$ on $M_{2^k}(\mathfrak{R})$. Suppose that $n \leq 2^k$. Let $e = \sum_{i=1}^n e_{ii}$ and

$$\nabla(a) = e\bar{\Delta}(a)e, a \in \bar{M}_n(\mathfrak{R}).$$

Then $\nabla : \bar{M}_n(\mathfrak{R}) \rightarrow \bar{M}_n(\mathfrak{R})$ and ∇ is a 2-local derivation on $\bar{M}_n(\mathfrak{R})$. Indeed, it is clear that ∇ is a map. At the same time, for all $a, b \in \bar{M}_n(\mathfrak{R})$ there exists a derivation $D : M_{2^k}(\mathfrak{R}) \rightarrow M_{2^k}(\mathfrak{R})$ such that

$$\bar{\Delta}(a) = D(a), \bar{\Delta}(b) = D(b).$$

Then

$$\nabla(a) = eD(a)e, \nabla(b) = eD(b)e.$$

By the proof of proposition 6 the map

$$\bar{D}(a) = eD(a)e, a \in \bar{M}_n(\mathfrak{R})$$

is a derivation and

$$\nabla(a) = \bar{D}(a), \nabla(b) = \bar{D}(b).$$

Hence, $\nabla : \bar{M}_n(\mathfrak{R}) \rightarrow \bar{M}_n(\mathfrak{R})$ is a 2-local derivation.

At the same time, on the subalgebra $\bar{M}_2(\mathfrak{R})$ the 2-local derivation ∇ coincides with the 2-local derivation Δ . Therefore, ∇ is an extension of Δ to $\bar{M}_n(\mathfrak{R})$. \triangleright

Proposition 9. *Let \mathfrak{R} be an associative unital ring, and let $M_n(\mathfrak{R})$, $n > 1$, be the matrix ring over \mathfrak{R} . Then, if every inner 2-local derivation on the matrix ring $M_n(\mathfrak{R})$ is an inner derivation then every inner 2-local derivation on the ring $\bar{M}_2(\mathfrak{R})$ is an inner derivation.*

Proof. Let Δ be a 2-local derivation on $\bar{M}_2(\mathfrak{R})$. Then by proposition 8 Δ is extended to a 2-local derivation $\bar{\Delta}$ on $M_n(\mathfrak{R})$. By the condition $\bar{\Delta}$ is an inner derivation, i.e. there exists $d \in M_n(\mathfrak{R})$ such that

$$\bar{\Delta}(a) = da - ad, a \in M_n(\mathfrak{R}).$$

But $\bar{\Delta}|_{\bar{M}_2(\mathfrak{R})} = \Delta$. Hence

$$\bar{\Delta}(a) = da - ad \in \bar{M}_2(\mathfrak{R})$$

for all $a \in \bar{M}_2(\mathfrak{R})$, i.e.

$$(e_{11} + e_{22})(da - ad)(e_{11} + e_{22}) = da - ad,$$

and $da - ad = ca - ac$ for all $a \in \bar{M}_2(\mathfrak{R})$, where $c = (e_{11} + e_{22})d(e_{11} + e_{22})$. Since $c \in \bar{M}_2(\mathfrak{R})$, we have that Δ is an inner derivation. The proof is complete. \triangleright

Proof of theorem 1. Propositions 4 and 9 immediately imply theorem1.

▷

We conclude the paper by the following more general observation.

Proposition 10. *Let $\Delta : \mathfrak{R} \rightarrow \mathfrak{R}$ be an inner 2-local derivation on an associative ring \mathfrak{R} . Suppose that \mathfrak{R} is generated by its two elements. Then, if Δ is additive then it is an inner derivation.*

Proof. Let x, y be generators of \mathfrak{R} , i.e. $\mathfrak{R} = \text{Alg}(\{x, y\})$, where $\text{Alg}(\{x, y\})$ is an associative ring, generated by the elements x, y in \mathfrak{R} . We have that there exists $d \in \mathfrak{R}$ such that

$$\Delta(x) = [d, x], \Delta(y) = [d, y],$$

where $[d, a] = da - ad$ for any $a \in \mathfrak{R}$.

Hence by the additivity of Δ we have

$$\Delta(x + y) = \Delta(x) + \Delta(y) = [d, x + y].$$

Note that

$$\begin{aligned} \Delta(xy) &= \Delta(x)y + x\Delta(y) = [d, x]y + x[d, y] = [d, xy], \\ \Delta(x^2) &= \Delta(x)x + x\Delta(x) = [d, x]x + x[d, x] = [d, x^2], \\ \Delta(y^2) &= \Delta(y)y + y\Delta(y) = [d, y]y + y[d, y] = [d, y^2], \end{aligned}$$

Similarly

$$\Delta(x^k) = [d, x^k], \Delta(y^m) = [d, y^m], \Delta(x^k y^m) = [d, x^k y^m]$$

and

$$\Delta(x^k y^m x^l) = \Delta(x^k y^m)x^l + x^k y^m \Delta(x^l) = [d, x^k y^m]x^l + x^k y^m [d, x^l] = [d, x^k y^m x^l].$$

Finally, for every polynomial $p(x_1, x_2, \dots, x_m) \in \mathfrak{R}$, where $x_1, x_2, \dots, x_m \in \{x, y\}$ we have

$$\Delta(p(x_1, x_2, \dots, x_m)) = [d, p(x_1, x_2, \dots, x_m)],$$

i.e. Δ is an inner derivation on \mathfrak{R} . ▷

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